

Dual WDVV Equations in $N = 2$ Supersymmetric Yang-Mills Theory

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Abstract

This paper studies the dual form of Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations in $N = 2$ supersymmetric Yang-Mills theory by applying a duality transformation to WDVV equations. The dual WDVV equations called in this paper are non-linear differential equations satisfied by dual prepotential and are found to have the same form with the original WDVV equations. However, in contrast with the case of weak coupling calculus, the perturbative part of dual prepotential itself does not satisfy the dual WDVV equations. Nevertheless, it is possible to show that the non-perturbative part of dual prepotential can be determined from dual WDVV equations, provided the perturbative part is given. As an example, the $SU(4)$ case is presented. The non-perturbative dual prepotential derived in this way is consistent to the dual prepotential obtained by D'Hoker and Phong.

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I. INTRODUCTION

We have had a rich harvest since the seed of approach using a Riemann surface for the low energy effective description of $N = 2$ supersymmetric Yang-Mills theory was sowed by Seiberg and Witten.^{1,2} For example, instanton effect³ for prepotentials obtained by using a Riemann surface⁴ showed a good agreement to the prediction of instanton calculus,⁵⁻⁷ integrable structure behind Seiberg-Witten solutions was discussed in terms of Whitham theory,⁸ and the approach taken by Seiberg and Witten was extended to the case of higher dimensional gauge theories.⁹ Of course, there are a lot of other interesting developments, but the best way to understand uniformly all these aspects of $N = 2$ supersymmetric gauge theories may be encoded in the language of Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations¹⁰⁻¹⁷ because they can widely cover various aspects of prepotentials.

In general, WDVV equations hold in various dimensional gauge theories^{15,18} and (in the case not including massive matter hypermultiplets) they are of the form

$$(\mathcal{F}_i)(\mathcal{F}_k)^{-1}(\mathcal{F}_j) = (\mathcal{F}_j)(\mathcal{F}_k)^{-1}(\mathcal{F}_i), \quad (1.1)$$

where \mathcal{F} is the prepotential, $(\mathcal{F}_i)_{jk} := \partial^3 \mathcal{F} / \partial a_i \partial a_j \partial a_k$ are matrix notations, a_i are regarded as periods of Seiberg-Witten differential and the indices run from 1 to the rank of the gauge group. (1.1) was extensively investigated at perturbative level,¹⁴⁻¹⁷ but not so much about the non-perturbative effect obtainable from (1.1) are known.¹⁹

On the other hand, the study of strong coupling region in view of WDVV equations are not found in literatures. According to the electro-magnetic duality of Seiberg and Witten,^{1,2} in the strong coupling region where charged particles become massless, the role of periods a_i and their magnetic duals $a_{Dj} := \partial \mathcal{F} / \partial a_j$ are exchanged. If this duality is applied to third-order derivatives of prepotential, we will obtain non-linear equations like (1.1) written in terms of dual periods. The equations obtained in this way will make it possible to derive the strong coupling prepotentials in the standpoint of WDVV equations. The aim of the paper is to study such non-linear equations.

This paper is organized as follows. In Sec. II, we apply the duality transformation to third-order derivatives of prepotential and by using (1.1) we derive non-linear equations for dual prepotential. These equations are found to have the same form with (1.1), so we call them as dual WDVV equations throughout the paper. In Sec. III, we consider the relation among dual prepotentials

and the dual WDVV equations. In particular, we firstly show in $SU(r+1)$ gauge theory that the dual perturbative prepotential^{20,21} do not satisfy dual WDVV equations. Of course, in order to determine non-perturbative part, the perturbative part must be required, so we provide it as input data. However, as the general case is slightly intractable, we present the calculation of non-perturbative part of dual prepotential in $SU(4)$ gauge theory as an example. We can find that the non-perturbative dual prepotential which is consistent to that found by D'Hoker and Phong²¹ is available from dual WDVV equations. Sec. IV is a brief summary.

II. THE DUAL WDVV EQUATIONS

In this section, we prove the existence of dual form of WDVV equations for all known models with WDVV equations (1.1). Our method here is based on the action of duality transformation for prepotential.

To begin with, let us consider how the third-order derivatives of prepotential transform under the electro-magnetic duality. In general, it is well-known that in the case of rank r gauge group the full electro-magnetic duality group is a subgroup of $Sp(2r, \mathbf{Z})$,^{22–27} and the generator

$$S := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (2.1)$$

where I and 0 are the unit matrix and zero matrix of size $r \times r$, respectively, induces the exchange of periods and their duals and therefore the inversion of the effective coupling constant. Note that in order to see a strong coupling behavior it is enough to take into account of only (2.1) and is not necessary to consider all actions of duality group.

In fact, the periods transform under (2.1) as

$$\begin{pmatrix} a_{D_i} \\ a_j \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{a}_{D_i} \\ \tilde{a}_j \end{pmatrix} := S \begin{pmatrix} a_{D_i} \\ a_j \end{pmatrix}. \quad (2.2)$$

Then the effective coupling constants transform as

$$\tau_{ij} := \frac{\partial a_{D_i}}{\partial a_j} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} \longrightarrow \tau_{D_{ij}} := \frac{\partial \tilde{a}_{D_i}}{\partial \tilde{a}_j} = -\frac{1}{\frac{\partial a_{D_j}}{\partial a_i}} = -\frac{1}{\tau_{ij}}, \quad (2.3)$$

where $\tau_{D_{ij}}$ are the dual effective coupling constants. From this transformation property, it is immediate to see that

$$\frac{\partial \tau_{D_{ij}}}{\partial \tilde{a}_k} = \sum_{l=1}^r \frac{\partial}{\partial a_l} \left(-\frac{1}{\tau_{ij}} \right) \frac{\partial a_l}{\partial \tilde{a}_k} = \sum_{l=1}^r \frac{(\mathcal{F}_i)_{jl}}{\tau_{ij}^2} \tau_{D_{lk}}. \quad (2.4)$$

Note that in (2.4) the repeated indices i and j are *not* summed.

Now, suppose that \tilde{a}_{D_i} are given by differentiations of some function \mathcal{F}_D (the dual prepotential)

$$\tilde{a}_{D_i} := \frac{\partial \mathcal{F}_D}{\partial \tilde{a}_i} = -\frac{\partial \mathcal{F}_D}{\partial a_{D_i}}. \quad (2.5)$$

Then the relation (2.4) can be rewritten as

$$(\mathcal{F}_{D_i})_{jk} = \sum_{l=1}^r (\mathcal{F}_i)_{jl} (\tau_D)_{lk}, \quad (2.6)$$

where

$$(\mathcal{F}_{D_i})_{jk} := \frac{\partial^3 \mathcal{F}_D}{\partial \tilde{a}_i \partial \tilde{a}_j \partial \tilde{a}_k} = -\frac{\partial^3 \mathcal{F}_D}{\partial a_{D_i} \partial a_{D_j} \partial a_{D_k}} \quad (2.7)$$

and $(\tau_D)_{lk} := \tau_{D_{lk}}$ are matrix notations and the overall factor is ignored because it is not necessary in the following discussions. Note that the right hand side of (2.6) is simply a matrix multiplication.

With the aid of (2.6), it is easy to see that

$$(\mathcal{F}_{D_i})_{pl} (\mathcal{F}_{D_k})_{lm}^{-1} (\mathcal{F}_{D_j})_{mn} = (\mathcal{F}_i)_{pl} (\mathcal{F}_k)_{lm}^{-1} (\mathcal{F}_j)_{mq} (\tau_D)_{qn}, \quad (2.8)$$

where $(\mathcal{F}_{D_k})_{lm}^{-1}$ mean the (l, m) components of $(\mathcal{F}_{D_k})^{-1}$ and we have ignored the determinants of (\mathcal{F}_k) and (τ_D) arising from $(\mathcal{F}_{D_k})^{-1}$ because they can be summarized into an overall factor. Thus from (2.8) and (1.1), we can obtain the dual form of WDVV equations

$$(\mathcal{F}_{D_i})(\mathcal{F}_{D_k})^{-1}(\mathcal{F}_{D_j}) = (\mathcal{F}_{D_j})(\mathcal{F}_{D_k})^{-1}(\mathcal{F}_{D_i}). \quad (2.9)$$

Since (2.9) is written by dual variables, we often refer (2.9) as dual WDVV equations throughout the paper.

From our construction, it would be obvious that there also exist dual WDVV equations if WDVV equations (1.1) hold.

III. DUAL PREPOTENTIAL AVAILABLE FROM DUAL WDVV EQUATIONS

The dual WDVV equations (2.9) take the same form with (1.1), but the study of dual prepotential in strong coupling region from a standpoint of (2.9) is slightly different from that in weak coupling calculus.

To see this, firstly, let us recall the prepotentials in weak coupling region. In this case, the WDVV equations were satisfied even at perturbative level.^{14,15,17,18} Namely, the perturbative prepotentials can be obtained by solving WDVV equations at perturbative level as was explicitly shown by Braden *et al.*¹⁸ in the case of $SU(4)$ gauge theory.

A. Perturbative dual prepotential of $SU(r+1)$ gauge theory

In the case of strongly coupled theory, on the other hand, the dual perturbative prepotentials themselves do not satisfy (2.9), thus in this case the dual WDVV equations do not hold at perturbative level.

To check this, let us recall the dual prepotential of $SU(r+1)$ gauge theory obtained from study of period integrals.^{20,21} According to the result, the perturbative part of dual prepotential $\mathcal{F}_{D,\text{per}}$ can be represented by a single function f

$$\mathcal{F}_{D,\text{per}} = \sum_{i=1}^r f(a_{D_i}). \quad (3.1)$$

As the argument of f is single, the matrices (\mathcal{F}_{D_i}) in (2.9) become singular, e.g., the only non-zero entry of (\mathcal{F}_{D_1}) is $(\mathcal{F}_{D_1})_{11} = \partial^3 \mathcal{F}_{D,\text{per}} / \partial a_{D_1}^3$. This indicates that we can not determine $\mathcal{F}_{D,\text{per}}$ from (2.9) even if we follow the method of Braden *et al.*¹⁸

B. Non-perturbative dual prepotential of $SU(4)$ gauge theory

Then, what happens when the non-perturbative part is introduced? In this case, we add the non-perturbative part $\mathcal{F}_{D,\text{non}}$ to $\mathcal{F}_{D,\text{per}}$ and consider

$$\mathcal{F}_D = \mathcal{F}_{D,\text{per}} + \mathcal{F}_{D,\text{non}}, \quad (3.2)$$

where

$$\mathcal{F}_{D,\text{non}} = \sum_{k=1}^{\infty} \mathcal{F}_{D,k} \Lambda^k \quad (3.3)$$

and $\Lambda^{-1} \equiv \Lambda_{\text{SU}(r+1)}$ is the dynamically generated mass scale of $\text{SU}(r+1)$ gauge theory. In (3.3), the coefficients $\mathcal{F}_{D,k}$ are functions in dual variables a_{D_i} .

As it is not easy to study the general case of r , we restrict $r = 3$ case in the following discussion. In this case, substituting (3.2) into (2.9), we can obtain nothing from the coefficient of Λ^0 , but we can find from the coefficient of Λ^1

$$f'''(a_{D_1})f'''(a_{D_2})f'''(a_{D_3})\partial_1\partial_2\partial_3\mathcal{F}_{D,1} = 0 \quad (3.4)$$

and from that of Λ^2

$$\begin{aligned} & f'''(a_{D_1})f'''(a_{D_2})\left(\partial_2\partial_3^2\mathcal{F}_{D,1} \cdot \partial_1\partial_3^2\mathcal{F}_{D,1} - \partial_3^3\mathcal{F}_{D,1} \cdot \partial_1\partial_2\partial_3\mathcal{F}_{D,1}\right) \\ & - f'''(a_{D_1})f'''(a_{D_3})\left(\partial_2^3\mathcal{F}_{D,1} \cdot \partial_1\partial_2\partial_3\mathcal{F}_{D,1} - \partial_2^2\partial_3\mathcal{F}_{D,1} \cdot \partial_1\partial_2^2\mathcal{F}_{D,1}\right) \\ & + f'''(a_{D_2})f'''(a_{D_3})\left(\partial_1^2\partial_3\mathcal{F}_{D,1} \cdot \partial_1^2\partial_2\mathcal{F}_{D,1} - \partial_1^3\mathcal{F}_{D,1} \cdot \partial_1\partial_2\partial_3\mathcal{F}_{D,1}\right) \\ & - f'''(a_{D_1})f'''(a_{D_2})f'''(a_{D_3})\partial_1\partial_2\partial_3\mathcal{F}_{D,2} = 0, \end{aligned} \quad (3.5)$$

where $f'''(a_{D_i}) = d^3f(a_{D_i})/da_{D_i}^3$ and $\partial_i \equiv \partial/\partial a_{D_i}$. It is interesting to notice that in the weak coupling calculus of $\text{SU}(4)$ gauge theory one-instanton prepotential satisfies a complicated equation¹⁹ while in the present case the equation for $\mathcal{F}_{D,1}$ is very simple.

To calculate $\mathcal{F}_{D,k}$ explicitly, the perturbative prepotential must be fixed, although it is not available directly from (2.9) in contrast with the weak coupling study. For this reason, we must provide it as the input data. Actually, the perturbative part is known to be calculated as^{20,21}

$$f(a_{D_i}) = a_{D_i}^2 \ln \frac{a_{D_i}}{\Lambda_{\text{SU}(4)}}, \quad (3.6)$$

where we have ignored the overall numerical factor and the normalization of $\Lambda_{\text{SU}(4)}$ in (3.6). In this case, the third-order derivatives of (3.6) do not vanish, so the general solution to (3.4) is easily calculated to give

$$\mathcal{F}_{D,1} = f_1(a_{D_2}, a_{D_3}) + f_2(a_{D_1}, a_{D_3}) + f_3(a_{D_1}, a_{D_2}), \quad (3.7)$$

where f_i are arbitrary functions.

For $\mathcal{F}_{D,2}$, on the other hand, from (3.5) and (3.7), we get

$$\begin{aligned}\mathcal{F}_{D,2} = & \frac{1}{2} \int (a_{D_1} \partial_1^2 \partial_3 f_2 \cdot \partial_1^2 \partial_2 f_3 + a_{D_2} \partial_1 \partial_2^2 f_3 \cdot \partial_2^2 \partial_3 f_1 + a_{D_3} \partial_2 \partial_3^2 f_1 \cdot \partial_1 \partial_3^2 f_2) da_{D_1} da_{D_2} da_{D_3} \\ & + g_1(a_{D_2}, a_{D_3}) + g_2(a_{D_1}, a_{D_3}) + g_3(a_{D_1}, a_{D_2}),\end{aligned}\quad (3.8)$$

where we have again used arbitrary functions g_i .

Here, let us notice that the scaling relation for $\mathcal{F}_{D,k}$ is given by

$$\sum_{i=1}^3 a_{D_i} \frac{\partial \mathcal{F}_{D,k}}{\partial a_{D_i}} = (2+k) \mathcal{F}_{D,k} \quad (3.9)$$

which follows from dimensional analysis. The scaling relation^{28–32} for dual prepotential^{21,33,34} was a basic tool of the study of strong coupling expansion presented by D'Hoker and Phong.²¹

Of course, though there are various functions satisfying (3.9), we can easily see that all monomials of degree 3 for (3.7) are also solutions to (3.9) by following to the method presented in weak coupling study of WDVV equations,¹⁹ thus we get

$$\mathcal{F}_{D,1} = \sum_{i=1}^3 s_i a_{D_i}^3 + c_1 a_{D_1}^2 a_{D_2} + c_2 a_{D_1}^2 a_{D_3} + c_3 a_{D_2}^2 a_{D_3} + c_4 a_{D_1} a_{D_2}^2 + c_5 a_{D_1} a_{D_3}^2 + c_6 a_{D_2} a_{D_3}^2, \quad (3.10)$$

where s_i and c_i are integration constants. The function form of (3.10) is consistent to the result of SU(4) gauge theory obtained by D'Hoker and Phong,²¹ but note that the integration constants should be determined by other approaches. A priori, we do not know explicit values for them in view of (2.9).

In a similar manner, we can determine $\mathcal{F}_{D,2}$ from (3.8) as

$$\begin{aligned}\mathcal{F}_{D,2} = & \sum_{i=1}^3 t_i a_{D_i}^4 + c_1 c_2 a_{D_1}^2 a_{D_2} a_{D_3} + c_3 c_4 a_{D_1} a_{D_2}^2 a_{D_3} + c_5 c_6 a_{D_1} a_{D_2} a_{D_3}^2 \\ & + \left\{ a_{D_1} a_{D_2}^3, a_{D_1}^2 a_{D_2}^2, a_{D_1}^3 a_{D_2}, a_{D_1} a_{D_3}^3, a_{D_1}^2 a_{D_3}^2, a_{D_1}^3 a_{D_3}, a_{D_2} a_{D_3}^3, a_{D_2}^2 a_{D_3}^2, a_{D_2}^3 a_{D_3} \right\},\end{aligned}\quad (3.11)$$

where t_i are integration constants, the braces mean any linear combination of the elements and we have assumed that g_i consist of polynomials.

Remark: In general, it is known from explicit examples^{4,33,35} that $\mathcal{F}_{D,k}$ are represented by polynomials in dual periods.

IV. SUMMARY

In this paper, we have considered the consequence of electro-magnetic duality transformation for the WDVV equations (1.1) and derived the dual WDVV equations (2.9) satisfied by dual prepotentials. The dual WDVV equations are turned out to have the same form with the original WDVV

equations, but the perturbative part of dual prepotential do not satisfy dual WDVV equations. However, we have derived the non-perturbative dual prepotential in pure $SU(4)$ gauge theory as an example by appropriately introducing perturbative part and following to the method to get solutions developed in weak coupling calculus.¹⁹ In fact, we have found that there is the non-perturbative prepotential in strong coupling region which is consistent to the result of D'Hoker and Phong.²¹ From this result, it is important to notice that we can study both weak and strong coupling prepotentials in the standpoint of WDVV equations.

As for an another direction to study the strong coupling region, it may be interesting to try to develop the topological string theoretic interpretation³⁶ in strong coupling region and to search a connection to (2.9). More detailed study should be expected in the future.

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